Radiography and Tomography in Linear Algebra

Possible Course Linear Algebra

Application Consider the common scenario of computerized axial tomography (CAT) scans of a human head. The CAT scan machine does not have a magic window into the body. Instead, it relies on sophisticated mathematical algorithms in order to interpret x-ray data. The x-ray machine produces a series (perhaps hundreds or thousands) of radiographs such as those shown in Figure 1, where each image is taken at a different orientation. Such a set of 2D images, while visually interesting and suggestive of many head features, does not directly provide a 3D view of the head. A 3D view of the head could be represented as a set of head slices (see figure 2), which when stacked in layers, provide a full 3D description. Our task is to produce a 3-dimensional image of a human head from a set of 2-dimensional radiographs.

Motivated Concepts Vector spaces, span, basis, linear transformations, null space, column space, the rank-nullity theorem, injectivity, surjectivity, invertibility, left-inverse, orthogonal matrices. Future labs will motivate the advanced concepts of pseudoinverse, matrix factorization, singular value decomposition and topics in ill-posed inverse problems.

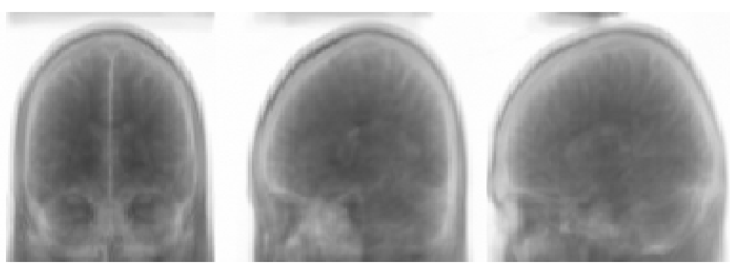

Figure 1: Example radiographs of a human head.

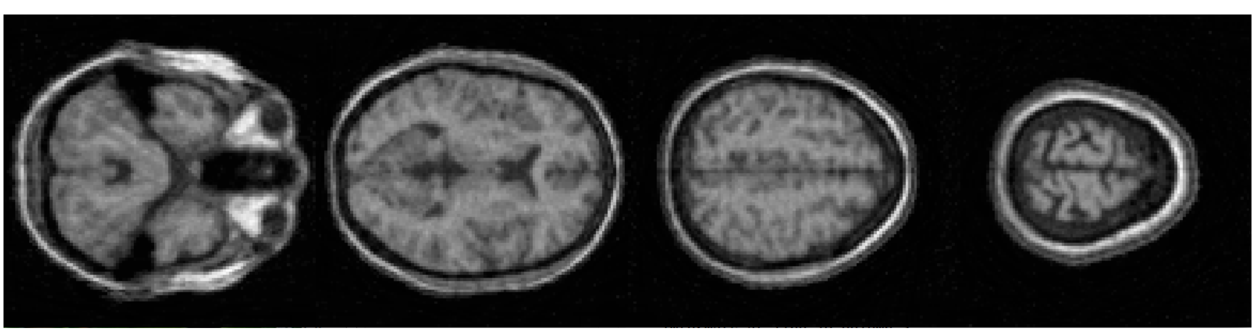

Figure 2: Example 2d slices of a 3d reconstruction.

Prerequisite Material Other prerequisites and material presentation suggestions are described individually for each Lab.

Description In this module students will be introduced to radiography and tomography as a physical application of linear algebra. The module will begin by introducing the space of images as a vector space. Students will explore the mathematics behind taking radiographs, or imaging an object onto a lower-dimensional subspace. They will also consider the inverse: given the radiograph, can one "reconstruct" the original object? Many early concepts – such as vector space, linear independence, span, basis, null space, range – appear naturally as a consequence of exploring properties of objects and their radiographs. When an operator is not invertible, students will explore the existence of a one-sided inverse and use this to obtain a 3D brain image from radiographs.

Future labs will have students explore more detailed "pseudo-inverse" operators obtained through the singular value decomposition of a matrix. They will use this to provide better approximate reconstructions of objects. They will also extend the ideas developed in the module to creatively explore how to leverage prior information about the object and how to deal with noisy data.

Time Frame The labs in this module typically take 3-5 weeks of class time, depending on how much of the material instructors will incorporate. They would be interleaved throughout the course. In conjunction with the Heat Diffusion Module, this module inspires most of the key concepts in a first course in Linear Algebra and many concepts found in a second course.

Labs 1 through 5 constitute the base material leading up to the students finding their first object reconstructions from radiographic data. In the process, they discover many of the key concepts pertaining to vector spaces and linear transformations. Future labs will develop advanced concepts associated with a pseudo inverse and singular value decomposition and allow the students to obtain dramatically improved object reconstructions.

Lab 1 Students explore the idea of an image as a member of a set of objects and how one naturally performs arithmetic on images. This exploration motivates ideas of vector, vector space properties, linear combination, span, and linear independence. No prior material from linear algebra is required. This lab makes a good introduction to vector spaces.

Lab 2 Students will explore the properties of linear transformations through constructing and using small examples of radiographic operators. Students should have a brief introduction to the physics of radiography before this lab. A summary document is provided for this purpose as supplementary reading. They should be very comfortable with matrix-vector multiplication.

Lab 3 This lab introduces injectivity, surjectivity and the null space of linear transformations in the context of radiographs and objects being radiographed. Students should *not* have proved theorems in this area yet, since the main idea is to get them to explore and discover some of the key ideas surrounding these concepts. This lab leads very naturally to the Rank-Nullity Theorem.

Lab 4 Students investigate properties of radiographic transformations. This lab can be done as a self-directed exercise or homework if the class has already discussed invertible transformations. It can also serve as a review of important concepts just before Lab 5.

Lab 5 In this lab students explore one way in which they might go about reconstructing the original 3D object from a radiograph when the radiographic transformation is not invertible. Specifically, they create a left-inverse of the radiographic transformation. This motivates one-sided inverses. In their reconstruction, they may also notice some flaws inherent to the process, which can be used as motivation for the use of the singular value decomposition in the next lab. Upon completion of this lab students will have performed their first image reconstruction.

Students should be already familiar with properties of symmetric matrices and relationships between rank, column space and null space of a matrix and its transpose.

Lab 6 Students will explore how to find pseudoinverses which serve as approximate left inverses. This exploration leads to the concept of the singular value decomposistion (SVD) of a transformation. Pseudoinverses can be radiograph-noise tolerant and exist for non-injective transformations. Students will build on their knowledge of diagonalization and orthogonal matrices.

Lab 7 Students will explore the use of the null space of a radiographic operator to enhance reconstructions based on object prior knowledge. This method can lead to dramatic improvements in reconstructions from limited radiographic data. Prerequisite material includes a firm understanding of Lab 6 and working with projections onto subspaces.